Discontinuous homomorphisms without Hamel bases

Paul Larson

Department of Mathematics
Miami University
Oxford, Ohio 45056

larsonpb@miamioh.edu

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joint work with Saharon Shelah

The Axiom of Choice

Two equivalent formulations of the Axiom of Choice:

- If $I \neq \emptyset$ and $A_i \neq \emptyset$ for each $i \in I$, then there is a function

$$f\colon I \to \bigcup_{i \in I} A_i$$

  such that $f(i) \in A_i$ for each $i \in I$.

- Every set can be wellordered.

Two consequences of Choice

- $\mathbb{R}$ can be wellordered.
- There is a function choosing a member from each $\mathbb{R}/\mathbb{Q}$ class. (A Vitali set, giving rise to a nonmeasurable set of reals.)

Hamel bases

A Hamel basis is a basis for $\mathbb{R}$ over the scalar field $\mathbb{Q}$.

Equivalently, it is a set $X \subseteq \mathbb{R}$ such that every element of $\mathbb{R}$ can be written as a linear combination of members of X (with rational coefficients) in exactly one way.

If $\mathbb{R}$ can be wellordered then Hamel bases exist.

Discontinuous homomorphisms

A function $f : \mathbb{R} \to \mathbb{R}$ is a *homomorphism* if

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$.

A continuous homomorphism has the form $f(x) = ax$, for some $a \in \mathbb{R}$.

A function is *measurable* if the preimage of each open interval is Lebesgue measurable.

It is a classical fact that every measurable homomorphism f from $\mathbb{R}$ to $\mathbb{R}$ is continuous.

However any (nontrivial) permutation of a Hamel basis extends to a (discontinuous) homomorphism.

Geometric Set Theory

Two joint results with Jindrich Zapletal:

- If there is a discontinuous homomorphism from $\mathbb{R}$ to $\mathbb{R}$, then there is a Vitali set.
- The existence of a Vitali set does not imply the existence of a discontinuous homomorphism (assuming the consistent existence of a strongly inaccessible cardinal).

The main theorem

The existence of a discontinuous homomorphisms does not imply the existence of Hamel basis.

Theorem (Larson-Shelah) If ZF is consistent, then so is ZF + "there exists a discontinuous homomorphism from $\mathbb{R}$ to $\mathbb{R}$" + "there is no Hamel basis for $\mathbb{R}$".

Forcing (1)

Suppose that M is a model of ZFC, and $(P, \leq)$ is a partial order in M.

A set $D \subseteq P$ is is *dense* in P if for each $p \in P$ there is a $q \in D$ with $q \leq p$.

A set $G \subseteq P$ is a *filter* if any two elements of G have a lower bound in G.

Forcing (2)

If $G \subseteq P$ intersects every dense subset of P in M then $M[G]$ is the smallest model of ZFC containing $M \cup \{G\}$.

Every element of $M[G]$ is the realization (by G) of a name in M.

Moreover, every statement that holds in $M[G]$ is forced to hold by some $p \in G$, meaning that the statement holds in every $M[H]$ with $p \in H$.

Partial orders

Two partial orders:

- Cohen forcing. $2^{<\omega}$, with $p \leq q$ if p extends q; adds a new function from ω to 2 (not in any meager set from the ground model).

- Random forcing. The non-null Borel sets under the order

$$p \leq q \Leftrightarrow \lambda(p \setminus q) = 0,$$

where λ denotes Lebesgue measure; adds a new real number (not in any null set from the ground model).

The consistent failure of Choice (1)

Suppose that M is a model of ZFC, and P is a partial order adding a new real number.

Let P_ω be the set of finitely-supported functions from ω to P, under the order of coordinatewise extension.

P_ω adds a sequence of new real numbers $\langle g_n : n \in \omega \rangle$.

The consistent failure of Choice (2)

Inside the model $M[\langle g_n : n \in \omega \rangle]$ we can form the model $M[\{g_n : n \in \omega\}]$.

This is a model of ZF for which every set is definable from an element of M, an ordinal, and finitely many members of $\{g_n : n \in \omega\}$.

Suppose toward a contradiction that $M[\{g_n : n \in \omega\}]$ contains a well-ordering of its real numbers.

This wellordering is then defined in this model by a formula $\varphi(x, y, \alpha, g_0, \ldots, g_k)$, for some k.

The consistent failure of Choice (3)

Some condition p in the generic filter then forces, for some ℓ, that g_ℓ is minimal (according to this wellorder) among the members of $\{g_n : n > k\}$.

Extend p to a condition p' (also in the generic filter) such that, for some $\ell' > \ell$, $p'(\ell) = p'(\ell')$.

Let $g'_n = g_n$ for $n \notin \{\ell, \ell'\}$, $g'_\ell = g_{\ell'}$ and $g'_{\ell'} = g_\ell$.

We have then that $M[\{g_n : n \in \omega\}]$ and $M[\{g'_n : n \in \omega\}]$ are the same model, but also that they disagree about the truth value of $\varphi(g_\ell, g_{\ell'}, \alpha, g_0, \ldots, g_k)$, a contradiction.

Partial homomorphisms

Suppose that $(G, +)$ is an abelian group.

Say that $h\colon G \to G$ is a partial homomorphism if the domain of h is a subgroup and

$$h(x + y) = h(x) + h(y)$$

for all $x, y \in \mathrm{dom}(h)$.

Amalgamating homomorphisms (1)

Given partial homomorphisms $h_1, \ldots, h_n$, there is a partial homomorphism h extending them all if and only if

$$h_1(x_1) + \cdots h_n(x_n) = h_1(y_1) + \cdots + h_n(y_n)$$

whenever

$$x_1 + \cdots + x_n = y_1 + \cdots y_n.$$

When $n = 2$, this happens if and only if h_1 and h_2 agree on the intersection of their domains.

Amalgamating homomorphisms (2)

When $n = 3$, this happens if and only if

$$h_1(x_1) + h_2(x_2) = h_3(x_3)$$

whenever $x_1 + x_2 = x_3$.

Extending homomorphisms

The additive group of $\mathbb{R}$ is divisible (meaning that x/n exists for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$).

It follows that any partial homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}, +)$ can be extended to a total homomorphism.

Mutually generic reals

Real numbers x_1 and x_2 are mutually generic random reals over V if the are produced by forcing with $P \times P$, where P is random forcing.

In this case $V[x_1] \cap V[x_2] = V$.

Furthermore, if $x_3 = x_1 - x_2$, then x_1, x_2 and x_3 are pairwise mutually generic random reals, and $V[x_i, x_j]$ is the same model for all $i \neq j$ from $\{1, 2, 3\}$.

The key lemma

Lemma. Suppose that x_1 and x_2 are mutually generic random reals over V, and let $x_3 = x_1 - x_2$.

- If y is an element of $(\mathbb{R}^{V[x_1]} + \mathbb{R}^{V[x_2]}) \cap \mathbb{R}^{V[x_3]}$, then there exist $a, b \in \mathbb{R}^V$ such that $y = ax_3 + b$.
- Let
  - h_0 be a homomorphism from $(\mathbb{R}, +)$ to itself in V,
  - c be a real number in V, and,
  - for each $i \in \{1, 2, 3\}$, h_i be, in $V[x_i]$, a homomorphism from $(\mathbb{R}, +)^{V[x_i]}$ to itself extending h_0, with $h_i(dx_i) = cdx_i$ for all $d \in \mathbb{R}^V$.

  Then h_1, h_2 and h_3 can be amalgamated in $V[x_1, x_2]$.

Proof of the first part (step one)

The proof of the first part of the lemma starts by considering a condition $(E, F) \in P \times P$ and Borel functions r, s and t such that (E, F) forces that

$$t(g_1 - g_2) = r(g_1) + s(g2).$$

It suffices to consider the case where E and F are both subsets of some interval $(-\epsilon, \epsilon)$ of full measure.

Extending an additive function

The proof of the first part of the lemma uses the fact that if $f\colon (-\epsilon, \epsilon) \to \mathbb{R}$ is such that

$$f(x+y) = f(x) + f(y)$$

whenever $|x|, |y|, |x+y| < \epsilon$, then f extends to a homomorphism $g\colon \mathbb{R} \to \mathbb{R}$, where $g(x) = nf(x/n)$ for any n such that $|x/n| < \epsilon$.

It follows that if f is Borel then there is a real number a such that $f(x) = ax$ for all $x \in (-\epsilon, \epsilon)$.

For the second part, it suffices to show that

$$h_1(y_1) + h_2(y_2) = h_3(y_3)$$

whenever $y_1 + y_2 = y_3$.

Fixing such y_1, y_2 and y_3, we have from the first part of the lemma that $y_3 = ax_3 + b$, for some $a, b \in \mathbb{R}^V$. Then

$$\begin{aligned} h_3(y_3) &= h_3(ax_3 + b) \\ &= h_3(ax_3) + h_3(b) \\ &= acx_3 + h_0(b) \\ &= ac(x_1 - x_2) + h_0(b) \\ &= acx_1 + h_1(b) - acx_2 \\ &= h_1(ax_1 + b) - h_2(ax_2). \end{aligned}$$

We also have that

$$y_1 + y_2 = y_3 = ax_3 + b = a(x_1 - x_2) + b = (ax_1 + b) - ax_2.$$

Since h_1 and h_2 are extensions of h_0 existing in mutually generic extensions of V they can be amalgamated, which implies that

$$h_1(y_1) + h_2(y_2) = h_1(ax + b) - h_2(ax).$$

P_{ω_1}

For each $d \subseteq \omega_1$ let P_d be the countable support product of random forcing, indexed by the elements of d.

So a condition p in P_d has the form

$$\{B^p_\alpha : \alpha \in \mathrm{supp}(p)\},$$

where supp(p) is a countable subset of d and each B^p_α is a Borel subset of $\mathbb{R}$ of positive Lebesque measure.

Then $p_2 \leq p_1$ in P_d if and only if $\mathrm{supp}(p_1) \subseteq \mathrm{supp}(p_2)$ and $\lambda(B^{p_2}_\alpha \setminus B^{p_1}_\alpha) = 0$ for each $\alpha \in \mathrm{supp}(p_1)$.

We write G_d for a generic filter for P_d, and $\dot{g}_\alpha$ for the natural $P_{\{\alpha\}}$-name for random real added by $P_{\{\alpha\}}$.

Q

We let Q be a partial order whose conditions are pairs $(d, \dot{h})$ such that d is a countable subset of ω_1 and $\dot{h}$ is a P_d-name for a homomorphism from $(\mathbb{R}, +)$ to itself.

The order is: $(d_1, \dot{h}_1) \leq (d_0, \dot{h}_0)$ if $d_0 \subseteq d_1$ and $1_{P_{d_1}}$ forces that $\dot{h}_{0,G_{d_0}} \subseteq \dot{h}_{1,G_{d_1}}$.

Given $q \in Q$ we write d_q and $\dot{h}_q$ for the first and second coordinates of q, respectively.

$Q * P_{\omega_1}$

We let $\underset{\sim}{\dot{H}}$ be the Q-name for the P_{ω_1}-name $\{\dot{h} : (d, \dot{h}) \in K\}$, where K denotes the generic filter for Q.

We implicity work with the (dense) set of condition (q, p) for which $d_q = \mathrm{dom}(p)$.

The amalgamation facts discussed before give the following.

- For each $\alpha \in \omega_1$, the set of $q \in Q$ with $\alpha \in d_q$ is dense.
- Every descending ω-sequence in Q has a lower bound.
- $\dot{\underset{\sim}{H}}_K$ is, the in the Q-extension $V[K]$, a P_{ω_1}-name for a discontinuous homomorphism from $(\mathbb{R}^{V[K,G_{\omega_1}]}, +)$ to itself.
- Letting (K, G_{ω_1}) denote a V-generic filter for $Q * P_{\omega_1}$, every every element of $\mathbb{R}^{V[K,G_{\omega_1}]}$ is an element of $\mathbb{R}^{V[G_d]}$, for some countable $d \subseteq \omega_1$.

The main theorem

Theorem. Let $(S, +_S)$ be a Polish vector space over $\mathbb{Q}$, and let

$$\pi : \mathbb{R} \to S$$

be an injective additive function. In any forcing extension

$$V[K, G_{\omega_1}]$$

by $Q \times P_{\omega_1}$, the function $\dot{\underset{\sim}{H}}_{K,G_{\omega_1}}$ is a discontinuous homomorphism from $(\mathbb{R}, +)$ to itself, but there is no Hamel basis for $(S, +_G)$ in the model

$$V(\mathbb{R}^{V[G_{\omega_1}]}, \dot{\underset{\sim}{H}}_{K,G_{\omega_1}}).$$

Proof (1)

Suppose toward a contradiction that $\dot{\underset{\sim}{B}}$ is a Q-name for a P_{ω_1}-name for a Hamel basis for $(S, +_S)$ in $V(\mathbb{R}^{V[G_{\omega_1}]}, \dot{\underset{\sim}{H}}_{K,G_{\omega_1}})$.

We may fix a condition (p_0, q_0), a $v \in V$, a formula φ, a cardinal λ and $P_{d_{q_0}}$-names $\dot{r}_1, \ldots, \dot{r}_k$ for elements of $\mathbb{R}$ such that (q_0, p_0) forces that

$$\dot{\underset{\sim}{B}} = \{y \in S : V_\lambda[K, G_{\omega_1}] \models \varphi(y, v, \dot{\underset{\sim}{H}}_{K,G_{\omega_1}}, \dot{r}_{1,G_{d_{q_0}}}, \ldots, \dot{r}_{k,G_{d_{q_0}}})\}.$$

Proof (2)

Lemma. For each $(q, p) \leq (q_0, p_0)$, for each P_{d_q}-name τ for an element of S, the statement $\tau \in \dot{\underset{\sim}{B}}$ is decided by (q, p') for densely many $p' \in P_{d_q}$ below p.

Otherwise we can find conditions $(q_1, p_1), (q_2, p_2) \leq (q, p)$ such that

- $d_{q_1} \cap d_{q_2} = d_q$ and
- $p_1 \restriction q_d = p_2 \restriction q_d$

and a P_{d_q}-name τ such that (q_1, p_1) and (q_2, p_2) force oppisite truth values for the statement $\tau \in \dot{\underset{\sim}{B}}$.

This is impossible since (q_1, p_1) and (q_2, p_2) would also be compatible.

Proof (3)

We can then let, for each $q \leq q_0$, $\dot{B}^q$ be the P_{d_q}-name for the value of

$$\dot{\underset{\sim}{B}}_{K,G_{\omega_1}} \cap S^{V[G_{d_q}]},$$

for any $Q \times P_{\omega_1}$-generic filter (K, G_{ω_1}) containing (q_0, p_0).

Then p forces in P_{d_q} that the realization of $\dot{B}^q$ will be a maximal linearly independent subset of the $\mathbb{R}$ of the P_{d_q}-extension.

Proof (4)

Lemma. If $q, q' \leq q_0$ with $d_q = d_{q'} = d$ and G and G' are (respectively), V-generic filters for P_d with $p_1 \in G \cap G'$ such that

$$\mathbb{R}^{V[G]} = \mathbb{R}^{V[G']}$$

and $\dot{h}_{q,G} = \dot{h}_{q',G'}$, then $\dot{B}^q_G = \dot{B}^{q'}_{G'}$.

Proof (4)

Fix two ordinals, $\alpha_1 < \alpha_2$ in $\omega_1 \setminus d_{q_0}$ and let

$$d = d_{q_0} \cup \{\alpha_1, \alpha_2\}.$$

Let G_0 be V-generic for $P_{d_{q_0}}$.

Let x_1 and x_2 be mutually generic random reals over $V[G]$, and let $x_3 = x_1 - x_2$.

Let G be the generic filter for P_d induced by G_0 and (x_1, x_2) in coordinates (α_1, α_2).

Let G' be the generic filter for P_d induced by G_0 and (x_1, x_3) in coordinates (α_1, α_2)

Proof (5)

Fix a real number $c \in V[G_{d_{q_0}}]$, and for each $i \in \{1,2,3\}$ let

$$h_i \in V[G_{d_{q_0}}][x_i]$$

be a homomorphism from

$$(\mathbb{R},+)^{V[G_{d_{q_0}}][x_i]}$$

to itself extending $\dot{h}_{q_0,G_{d_{q_0}}}$, with

$$h_i(dx_i) = cdx_i$$

for all $d \in \mathbb{R}^{V[G_{d_{q_0}}]}$.

Applying the key lemma, let h be an amalgamation of h_1, h_2 and h_3 in $V[G_d]$.

Proof (6)

The lemma above implies that there is a Hamel basis B for S in $V[G_{d_{q_0}}][x_1, x_2]$ whose restriction to each of $V[G_{d_{q_0}}][x_1]$, $V[G_{d_{q_0}}][x_2]$ and $V[G_{d_{q_0}}][x_3]$ is in the corresponding model.

However, since x_1, x_2 and x_3 are pairwise mutually generic over $V[G_{d_{q_0}}]$, the intersection of any two of these models is $V[G_{d_{q_0}}]$.

If such a B did exist, then each of $\pi(x_1)$, $\pi(x_2)$ and $\pi(x_3)$ would be a linear combination of elements of the corresponding set $B \cap V[G_{d_{q_0}}][x_i]$ (with rational coefficients) in a unique way.

The equation $\pi(x_3) +_S \pi(x_2) = \pi(x_1)$ however then gives two different linear combinations for $\pi(x_1)$.

The construction above gives a homomorphism *h* which is surjective but not injective.

It can be modified in the following ways.

- Require *h* to be injective but not surjective. .
- Require *h* to be bijective.
- Require the range of *h* to be $(\mathbb{Q}, +)$.
- Require $\mathbb{R}/\{x : h(x) = x\}$ to have dimension 2.

Question (Zapletal) Does the existence of a Hamel basis for $\mathbb{R}^2$ imply the existence of a Hamel basis for $\mathbb{R}$?